

Skyrmions, multi-instantons and $SU(\infty)$ -Toda equation

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Abstract

We construct Skyrmions from holonomy of the spin connection of multi-Taub-NUT instantons with the centres positioned along a line in \mathbb{R}^3 . Our family of Skyrmions includes the Taub-NUT Skymion previously constructed by Dunajski. However, we demonstrate that different gauges of the spin connection can result in Skyrmions with different topological degrees. As a by-product, we present a method to compute the degrees of the Taub-NUT and Atiyah-Hitchin Skyrmions analytically; these degrees are well defined as a preferred gauge is fixed by the $SU(2)$ symmetry of the two metrics.

Regardless of the gauge, the domain of our Skyrmions is the space of orbits of the axial symmetry of the multi-Taub-NUT instantons. We obtain an expression for the induced metric on the space and its associated solution to the $SU(\infty)$ -Toda equation.

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1 Introduction

While the attempt to unify gravity into the same framework as that of quantum field theories of particles is still the most ambitious goal in theoretical physics, a geometric model of particles which uses Riemannian geometry rather than field theory has been proposed in [2]. In this paper, Atiyah, Manton and Schroers (AMS) proposed a description of elementary particles in terms of self-dual gravitational instantons. These are 4-dimensional Riemannian manifolds which are Einstein, having self-dual Weyl tensor, and whose curvature decays at infinity [10]. The topological invariants of the manifolds are identified with quantum numbers of particles. For example, the electrically charged particles are modelled by non-compact asymptotically locally flat (ALF) instantons, where the electric charge is given by the first Chern class of the asymptotic circle fibration. The baryon number is proposed to be the signature of the manifold.

The AMS model is inspired by an older geometric model of particles, called the Skyrme model [17]. This is a nonlinear field theory which can be regarded as an effective, low energy approximation of Quantum Chromodynamics in the limit of large number of quarks. In the Skyrme model, baryons are described by soliton solutions of the theory, called Skyrmions. A Skyrme field at a given time is a map $U : \mathbb{R}^3 \rightarrow SU(2)$, which satisfies the boundary condition $U(\mathbf{x}) \rightarrow \mathbf{1}$ as $\mathbf{x} \rightarrow \infty$. This boundary condition ensures that the topological degree of the map U is well-defined, and it is then identified with baryon numbers. The Euler-Lagrange equation of the Skyrme model is not integrable. However, good approximations of the solutions can be generated from $SU(2)$ Yang-Mills instantons in \mathbb{R}^4 . This is done by calculating the holonomy of the Yang-Mills instantons along lines in a fixed direction [1]. The topological degrees of the resulting Skyrmions are given by the instanton numbers of the corresponding Yang-Mills fields.

While it is shown in [2] how gravitational instantons describe some static particles, namely electron, proton, neutron and neutrino, and how quantum numbers can be interpreted as topological invariants, yet a construction of an associated Skyrmion had not been explored. This was later done in [6], where a construction of a Skyrmion from a holonomy of the self-dual spin connection of a gravitational instanton has been proposed. The gravitational instanton is assumed to be self-dual and Ricci-flat. The construction was applied to the Taub-NUT and Atiyah-Hitchin instantons, which in the AMS model describe electron and proton, respectively.

Motivated by the work in [6], this paper aims to explore further implementation of the construction in [6] to a family of gravitational instantons, namely the multi-Taub-NUT instantons [10]. The multi-Taub-NUT instantons form a family of ALF instantons, known as type A_{N-1} , where $N = 1$ corresponds to the Taub-NUT instanton. The multi-

Taub-NUT metric is known explicitly as

$$g = V(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)) + V^{-1}(d\psi + \alpha)^2, \quad V = 1 + \sum_{n=1}^N \frac{\varepsilon}{\|\mathbf{x} - \mathbf{x}_n\|}, \quad (1.1)$$

where $r \in [0, \infty)$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$ are the usual spherical coordinates, and $dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$ is the flat metric on \mathbb{R}^3 . The function $V(\mathbf{x})$ is a solution to the Laplace's equation with $\varepsilon > 0$ and \mathbf{x}_n denote N distinct points on \mathbb{R}^3 . The one-form α is related to V via $d\alpha = *_3 dV$, where $*_3$ is the Hodge star operator with respect to the flat metric.

Here, we shall only consider the multi-Taub-NUT instantons where the points \mathbf{x}_n , the NUTs, lie along the union of the lines $\theta = 0$ and $\theta = \pi$, i.e. the z -axis in the Cartesian coordinate system (x, y, z) for \mathbb{R}^3 . Such multi-Taub-NUTs have two generators of Killing symmetry, one is the vector field $\frac{\partial}{\partial\psi}$ and the other $\frac{\partial}{\partial\phi}$. We shall only use the Killing vector field $\frac{\partial}{\partial\phi}$ for the Skymion construction, as it gives rise to the Skymion with nonzero topological degree in the case of Taub-NUT instanton [6].

Following the construction in [6], we shall obtain the $SU(2)$ Yang-Mills connection on the multi-Taub-NUT background from the spin connection of the multi-Taub-NUT itself. Such procedure was first introduced in [3], and further investigated in [4, 16, 13]. The holonomy of the Yang-Mills connection will then be calculated along the orbits of the axial symmetry of the multi-Taub-NUT instanton. This will give rise to an $SU(2)$ -valued scalar field U - our Skymion - on the space of orbits of the Killing symmetry.

In the next section we shall review the construction [6] of Skymions from gravitational instantons, and apply it to the multi-Taub-NUT instantons. We choose the frame fields for the self-dual spin connection to be a natural extension of the $SU(2)$ -invariant frame fields used in [6]. We then obtain an explicit expression for the multi-Taub-NUT Skymions, in term of V and α in (1.1); the family includes the Taub-NUT Skymion previously constructed in [6] as the case $N = 1$. In Section 3 we shall however show that different gauges of the spin connection can result in Skymions with unequal topological degrees. In particular, writing the multi-Taub-NUT spin connection in a different set of frame fields, we show that the resulting family of Skymions has vanishing topological degree for all $N \geq 1$. Unlike the Taub-NUT and Atiyah-Hitchin Skymions where the $SU(2)$ symmetry is present, it is not obvious how one can justify a preferred gauge for the multi-Taub-NUT $N \geq 2$ case. Nevertheless, as a by-product, we present a method to compute the topological degrees of the Taub-NUT and Atiyah-Hitchin Skymions analytically. (These degrees are well defined as they are fixed by the frame fields chosen to respect the $SU(2)$ symmetry of both metrics.) Comparing with the degrees obtained

in [6] by the method of preimages counting, our procedure yields the same result for the Taub-NUT case. The degree of the Atiyah-Hitchin Skyrmion agrees with that in [6] up to a sign.

The space where the Skyrmons live is the space of orbits of the axial symmetry of the multi-Taub-NUT metric. In Section 4 we investigate the Einstein-Weyl metric on the space. In particular, we obtain implicit expressions for the metric and its associated solution to the $SU(\infty)$ -Toda equation.

2 Skyrmion construction

The Skyrmion construction of [6] is based on two important results. One is that a solution of the self-dual Yang-Mills equation on a Ricci-flat background can be obtained from the spin connection of the background metric itself [3]. The other is that static Skyrmons can be generated from $SU(2)$ Yang-Mills instantons [1].

2.1 From spin connection to Yang-Mills instanton

Let (M, g) be a Riemannian four manifold with a Ricci-flat metric g , and ω_{ab} be the connection one-form defined from an orthonormal tetrad of one-forms $\{e_a\}$ by

$$de_a = \omega_{ab} \wedge e_b, \quad \omega_{(ab)} = 0, \quad a, b = 0, 1, 2, 3.$$

Charap and Duff showed that one can identify ω^{ab} with an $O(4)$ Yang-Mills potential, which under the decomposition $O(4) = SU(2) \times SU(2)$ gives rise to an $SU(2)$ Yang-Mills potential satisfying the (anti-) self-dual equation on the metric background.

Recall [5] that the complexified tangent bundle can be decomposed as $TM \otimes \mathbb{C} \cong S_+ \otimes S_-$, where S_{\pm} are rank two complex vector bundles. The spin bundles S_{\pm} inherit connections γ_{\pm} (spin connections) from the Levi-Civita connection of g . Let us choose a convention such that γ_+ encodes the information about the self-dual part of the Weyl tensor, and γ_- the anti-self-dual part. Then it means that given a self-dual Ricci-flat metric, one can interpret the spin connection γ_+ as an $SU(2)$ self-dual Yang-Mills connection on the metric background.

Pope and Yuille applied such procedure to construct an $SU(2)$ self-dual Yang-Mills instanton in the Taub-NUT background [16]. The result was then used in one of the examples in [6] to generate a Skyrmion from a gravitational instanton, namely the Taub-NUT Skyrmion. The goal of our paper is to extend the Skyrmion construction in [6] to multi-Taub-NUT instantons. Let us now describe how one can define a Yang-Mills instanton from the spin connection of a multi-Taub-NUT metric.

The multi-Taub-NUT metrics form a family of gravitational instantons which are hyperKähler, thus they are Ricci-flat and have self-dual Riemann tensor. They are asymptotically flat (ALF), meaning the metrics approach S^1 bundles over S^2 at infinity. The metrics of the multi-Taub-NUT family, also called the type A_{N-1} , are given explicitly by the Gibbons-Hawking ansatz [9] (1.1). They have a triholomorphic S^1 symmetry, generated by the vector field $\frac{\partial}{\partial\psi}$. For the purpose of our Skyrmion construction, we shall only consider a subclass of the multi-Taub-NUT instantons, where the points \mathbf{x}_n , the NUTs, lie along the z -axis in the Cartesian coordinate system. This results in the function V being independent of ϕ . Moreover, we shall set $\varepsilon = 1$, thus the range of ψ coordinate is $[0, 4\pi)$.

The self-dual spin connection, $\gamma := \gamma_+$, of a multi-Taub-NUT metric can be calculated using the self-dual two-forms

$$\Sigma_i = e_0 \wedge e_i + \frac{1}{2} \varepsilon_{ijk} e_i \wedge e_j, \quad i, j, k = 1, 2, 3, \quad (2.2)$$

where $\{e_0, e_i\}$ is an orthonormal tetrad of one-forms. The spin connection coefficients, γ_{ij} , are determined from

$$d\Sigma_i + \gamma_{ij} \wedge \Sigma_j = 0. \quad (2.3)$$

Consider the metric of the form in (1.1)

$$g = V(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)) + V^{-1}(d\psi + \alpha)^2, \quad (2.4)$$

where V is now independent of ϕ and is given by

$$V(r, \theta) = 1 + \sum_{n=1}^N \frac{1}{\sqrt{r^2 - 2z_n r \cos\theta + z_n^2}}, \quad (2.5)$$

and the one-form α is determined from $*_3 dV = d\alpha$ to be

$$\alpha = \hat{\alpha}(r, \theta) d\phi, \quad \text{where} \quad \hat{\alpha}(r, \theta) = \sum_{n=1}^N \frac{r \cos\theta - z_n}{\sqrt{r^2 - 2z_n r \cos\theta + z_n^2}}. \quad (2.6)$$

Now, let

$$\begin{aligned} \eta_1 &= -\sin\psi d\theta + \sin\theta \cos\psi d\phi, \\ \eta_2 &= \cos\psi d\theta + \sin\theta \sin\psi d\phi, \\ \eta_3 &= d\psi + \hat{\alpha} d\phi. \end{aligned} \quad (2.7)$$

Then (2.4) becomes

$$g = V(dr^2 + r^2(\eta_1^2 + \eta_2^2)) + V^{-1}\eta_3^2.$$

We can then choose an orthonormal tetrad of one-forms to be

$$e_0 = \sqrt{V} dr, \quad e_1 = r\sqrt{V} \eta_1, \quad e_2 = r\sqrt{V} \eta_2, \quad e_3 = \frac{1}{\sqrt{V}} \eta_3. \quad (2.8)$$

For the Taub-NUT metric, with $N = 1, z_1 = 0$, we have that $V = 1 + \frac{1}{r}$, $\hat{\alpha} = \cos \theta$ and (η_1, η_2, η_3) in (2.7) are left-invariant one-forms on $SU(2)$, which satisfy

$$d\eta_1 = \eta_2 \wedge \eta_3, \quad d\eta_2 = \eta_3 \wedge \eta_1, \quad d\eta_3 = \eta_1 \wedge \eta_2.$$

Thus the tetrad (2.8) is $SU(2)$ -left invariant.

For the multi-Taub-NUT, the metric no longer has spherical symmetry and (η_1, η_2, η_3) in (2.7) are not left-invariant one-forms on $SU(2)$. However, to obtain a family of Skyrmions which includes the Taub-NUT Skyrmion in [6], we shall proceed by using (2.8) as our orthonormal tetrad, with V and α given in (2.5, 2.6).

The spin connection coefficients, γ_{ij} , are determined from (2.3). Then the $SU(2)$ Yang-Mills potential, A , is given by

$$A = \frac{1}{2} \varepsilon_{ijk} \gamma_{jk} \otimes \mathbf{t}_i, \quad (2.9)$$

where $\{\mathbf{t}_i\}$ are generators of the Lie algebra $\mathfrak{su}(2)$ satisfying $[\mathbf{t}_i, \mathbf{t}_j] = -\varepsilon_{ijk} \mathbf{t}_k$.

That is,

$$A = A_1 \otimes \mathbf{t}_1 + A_2 \otimes \mathbf{t}_2 + A_3 \otimes \mathbf{t}_3,$$

where $A_1 = \gamma_{23}$, $A_2 = \gamma_{31}$ and $A_3 = \gamma_{12}$. We find that

$$\begin{aligned} A_1 &= \frac{\sin \psi}{rV \sin \theta} \hat{\alpha}_r dr - \left(1 + \frac{rV_r}{V}\right) \eta_1 - \frac{\cos \psi}{rV^2 \sin \theta} \hat{\alpha}_r \eta_3 \\ A_2 &= -\frac{\cos \psi}{rV \sin \theta} \hat{\alpha}_r dr - \left(1 + \frac{rV_r}{V}\right) \eta_2 - \frac{\sin \psi}{rV^2 \sin \theta} \hat{\alpha}_r \eta_3 \\ A_3 &= -\frac{\cos \psi}{V \sin \theta} (\hat{\alpha}_r + V(\hat{\alpha} - \cos \theta)) \eta_1 - \frac{\sin \psi}{V \sin \theta} (\hat{\alpha}_r + V(\hat{\alpha} - \cos \theta)) \eta_2 + \left(1 + \frac{V_r}{V^2}\right) \eta_3, \end{aligned} \quad (2.10)$$

which gives the result in [6] for the Taub-NUT case when $\hat{\alpha} = \cos \theta$ and $V = 1 + \frac{1}{r}$.

2.2 From Yang-Mills instantons to Skyrmons

Atiyah and Manton showed in [1] that Skyrme field configurations can be generated from $SU(2)$ Yang-Mills instantons. Given any such Yang-Mills field in \mathbb{R}^4 , the holonomy of the Yang-Mills field along all lines parallel to the time axis gives rise to a scalar $SU(2)$ -valued function U on \mathbb{R}^3 . It was shown that U satisfies the boundary condition $U(\mathbf{x}) \rightarrow \mathbf{1}$ as $\mathbf{x} \rightarrow \infty$ and thus can be regarded as a Skyrmon. Dunajski [6] applied the Atiyah and Manton construction to the Yang-Mills fields from gravitational instantons, namely the Taub-NUT and the Atiyah-Hitchin instantons. One difference is that the Yang-Mills instantons now live on curved backgrounds. It is noted that the holonomy should be calculated along orbits of a Killing vector field of the background manifold, so that the space of orbits \mathcal{B} , where the Skyrmon lives, admits a metric.

Suppose K is a Killing vector field and $x^a = (s, \mathbf{x})$ be coordinates on the manifold M such that $K = K^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial s}$. Then formally the holonomy is given by

$$U = \mathcal{P} \exp \left(- \int_{\Gamma} A \right), \quad (2.11)$$

where \mathcal{P} denotes s -ordering. In general, one computes U by solving the equation $K^a D_a \Psi = 0$, where $D_a = \partial_a + A_a$. For a non-compact Γ , one solves for a scalar $SU(2)$ -valued function $\Psi(s, \mathbf{x})$ under the initial condition $\Psi(-\infty, \mathbf{x}) = \mathbf{1}$. Then $U(\mathbf{x}) = \Psi(\infty, \mathbf{x})$. If the orbit Γ is S^1 , one needs to cut it into an interval. In [6], the holonomy is calculated along the S^1 orbits of a left translation $SO(2)$ inside $SU(2)$, generated by the vector field $\frac{\partial}{\partial \phi}$. The orbits generated by $\frac{\partial}{\partial \psi}$ was not considered as it gives an Abelian Skyrmon with zero topological degree.

We shall now follow [6] and choose the orbit Γ to be that generated by $\frac{\partial}{\partial \phi}$, which is a Killing vector field for our axially-symmetric multi-Taub-NUT instantons. Let the component A_ϕ be the restriction of A to the orbit Γ . Thus,

$$U = \mathcal{P} \exp \left(- \int_0^{2\pi} A_\phi d\phi \right). \quad (2.12)$$

Now, since all the spin connection coefficients (2.10) are independent of ϕ , the integral (2.12) can be evaluated explicitly. This gives

$$U = \exp(-2\pi A_\phi),$$

where $A_\phi = A_{1\phi} \otimes \mathbf{t}_1 + A_{2\phi} \otimes \mathbf{t}_2 + A_{3\phi} \otimes \mathbf{t}_3$, and $A_{j\phi} = \frac{\partial}{\partial \phi} \lrcorner A_j$.

Now, let $\gamma_1 := A_{1\phi}$, $\gamma_2 := A_{2\phi}$ and $\gamma_3 := A_{3\phi}$. From (2.7) and (2.10) we have

$$\begin{aligned}\gamma_1 &= -\cos\psi \left(\sin\theta + \frac{1}{r} \left(\frac{\hat{\alpha}}{V} \right)_\theta \right) \\ \gamma_2 &= -\sin\psi \left(\sin\theta + \frac{1}{r} \left(\frac{\hat{\alpha}}{V} \right)_\theta \right) \\ \gamma_3 &= \cos\theta - \left(\frac{\hat{\alpha}}{V} \right)_r,\end{aligned}\tag{2.13}$$

where we have simplified the expressions using the relation $*_3 dV = d\alpha$ in spherical coordinates:

$$\hat{\alpha}_r = -\sin\theta V_\theta, \quad \hat{\alpha}_\theta = r^2 \sin\theta V_r.$$

Suppose we choose the representation $\mathbf{t}_j = \frac{i}{2}\tau_j$, where τ_j are Pauli matrices. Then the Skyrmion from the axially-symmetric multi-Taub-NUT instanton is given by

$$U(r, \theta, \psi) = \exp(-i\pi\gamma_j\tau_j),\tag{2.14}$$

where γ_j are given in (2.13). The expression (2.14) simplifies in the case of the Taub-NUT instanton (see [6]). Also, using (2.13) it can be shown that the multi-Taub-NUT Skyrmion approaches a constant group element at infinity. In particular, like the Taub-NUT case, $U \rightarrow -\mathbf{1}$ as $r \rightarrow \infty$.

We note here that the S^1 orbits along which we calculate the holonomy have no point in common. Therefore, unlike in the Atiyah-Manton construction [1], the initial condition at the based point of all the orbits cannot be imposed. Thus gauge transformation of the Yang-Mills potential (2.9) will affect the resulting Skyrme field. To have a well-defined Skyrmion, a preferred gauge has to be chosen. In [6], the preferred gauge for the Taub-NUT and Atiyah-Hitchin instantons has been fixed by choosing the $SU(2)$ invariant frame fields (2.8). This choice of gauge is resulted from the symmetry requirement, that the Yang-Mills potential is $SU(2)$ left-invariant, and the requirement that the potential is regular at $r = 0$. The multi-Taub-NUT metrics for $N \geq 2$ do not have the $SU(2)$ symmetry, and it is not obvious how one can justify a preferred gauge. However, the family of Skyrmons (2.14) can be considered as a generalisation of the Taub-NUT Skyrmion of [6], as it includes the Skyrmion in the family.

3 The topological degree

The Skyrme field constructed in Section 2 is a map from the space of orbits \mathcal{B} of the Killing vector field to the Lie group $SU(2)$. An integral expression for a topological degree of the map $U : \mathcal{B} \rightarrow SU(2)$ is given by

$$D = -\frac{1}{24\pi^2} \int_{\mathcal{B}} \text{Tr}((U^{-1}dU)^3). \quad (3.15)$$

In [6], the topological degrees of the Taub-NUT and Atiyah-Hitchin Skyrmons are calculated. It is hoped that they can be interpreted as some physical quantum numbers of the particles modelled by the gravitational instantons, namely the electron and proton. Recall that the Taub-NUT and Atiyah-Hitchin Skyrmons are constructed in the gauge preferred by the $SU(2)$ symmetry, and their topological degrees are 2 and 1, respectively.

However, we shall show below that there exists a tetrad of frame fields different from (2.8) such that the resulting family of multi-Taub-NUT Skyrmons has vanishing topological degree for all $N \geq 1$. This finding may not present a problem to the physical interpretation of the topological degrees of the Taub-NUT ($N = 1$) and Atiyah-Hitchin Skyrmons, as the symmetry requirement fixes a preferred gauge in which the degrees are nonzero. However, the situation is different for the multi-Taub-NUT case. Although the choice of frame fields (2.8) resulting in our multi-Taub-NUT Skyrmons (2.14) is supported by the demand that the family should include the Taub-NUT Skyrmon of [6], it is not clear to the author if there is a symmetry or regularity requirement that would lead to the gauge when $N \geq 2$.

As a by-product of this study, we shall also present an analytic method to compute the topological degrees of the Taub-NUT and Atiyah-Hitchin Skyrmons, which were previously obtained in [6] by the method of preimages counting.

3.1 Gauge dependence

Here we shall demonstrate that there exists an orthonormal tetrad of one-forms such that the multi-Taub-NUT spin connection gives rise to the Skyrmons with zero topological degree for all $N \geq 1$. Since the multi-Taub-NUT instantons (2.4) have axial symmetry, instead of (2.8) one could think of another tetrad of one-forms which is perhaps more natural with regards to the isometry of the metrics. To define this tetrad, let us write the metric (2.4) in the cylindrical coordinates on \mathbb{R}^3 :

$$g = V(d\rho^2 + \rho^2 d\phi^2 + dz^2) + V^{-1}(d\psi + \alpha)^2,$$

where now

$$V = 1 + \sum_{n=1}^N \frac{1}{\sqrt{\rho^2 + (z - z_n)^2}},$$

$$\alpha = \hat{\alpha}(\rho, z) d\phi, \quad \text{where} \quad \hat{\alpha} = \sum_{n=1}^N \frac{z - z_n}{\sqrt{\rho^2 + (z - z_n)^2}}.$$

Then a natural tetrad of one-forms to the metric is given by

$$e'_0 = \frac{1}{\sqrt{V}} (d\psi + \cos \theta d\phi), \quad e'_1 = \sqrt{V} d\rho, \quad e'_2 = \rho \sqrt{V} d\phi, \quad e'_3 = \sqrt{V} dz. \quad (3.16)$$

Using this tetrad and repeating the same procedure in Section 2 yields a new Yang-Mills potential A' given by

$$\begin{aligned} A'_1 &= \left(\frac{1}{V} \right)_\rho d\psi + \left(\frac{\hat{\alpha}}{V} \right)_\rho d\phi, \\ A'_2 &= \frac{V_z}{V} d\rho - \frac{V_\rho}{V} dz \\ A'_3 &= \left(\frac{1}{V} \right)_z d\psi + \left(\left(\frac{\hat{\alpha}}{V} \right)_z - 1 \right) d\phi. \end{aligned} \quad (3.17)$$

Unlike the Yang-Mills field (2.10) defined in Section 2, the potential (3.17) is independent of the ψ coordinate. Thus the resulting Skyrme field U only depends on two coordinates of the three-dimensional space \mathcal{B} . Therefore the three-form $(U^{-1}dU)^3$ in the integral (3.15) vanishes and gives zero topological degree for all parameter $N \geq 1$.

It is well known that under the change of tetrads, the spin connection γ_{ij} and thus the Yang-Mills field A change under the usual gauge transformation. Hence, it demonstrates that different gauges of the spin connection result in Skymions with different topological degrees. This presents a problem to the interpretation of the topological degree as a quantum number of the system of N -electrons for $N \geq 2$.

3.2 Taub-NUT and Atiyah-Hitchin Skymions

We shall end Section 3 by presenting an analytic method to compute the topological degree (3.15) of the Taub-NUT and Atiyah-Hitchin Skymions, constructed in the gauge preferred by the $SU(2)$ symmetry.

First, one notes that the integrand of (3.15) is actually the pullback of the normalised volume form on $S^3 = SU(2)$. That is, suppose (Θ, Φ, Ψ) are the Euler angles such that a

point $\mathbf{q} \in S^3$ can be parametrised by

$$\mathbf{q} = \cos \Psi \mathbf{1} - i (\sin \Psi \sin \Theta \cos \Phi \tau_1 + \sin \Psi \sin \Theta \sin \Phi \tau_2 + \sin \Psi \cos \Theta \tau_3), \quad (3.18)$$

where $\Theta \in [0, \pi)$, $\Phi \in [0, 2\pi)$, $\Psi \in [0, \pi)$ and τ_j are Pauli matrices. Then

$$\Omega = \sin^2 \Psi \sin \Theta d\Psi \wedge d\Theta \wedge d\Phi$$

is a volume form on S^3 , and it can be shown that (3.15) is given by

$$D = -\frac{1}{2\pi^2} \int_{\mathcal{B}} {}^*\Omega,$$

where ${}^*\Omega$ denotes the pullback of Ω onto \mathcal{B} . Now, since Ω is closed, it can be written locally as $\Omega = d\hat{\omega}$ for some two-form $\hat{\omega}$ on S^3 . Moreover, as the exterior derivative d commutes with the pullback, by letting $\omega := {}^*\hat{\omega}$ one has that

$$D = -\frac{1}{2\pi^2} \int_{\mathcal{B}} d\omega. \quad (3.19)$$

The integral (3.19) can be evaluated using the asymptotic values of ω as follows.

Let $\omega = \omega_1 dr \wedge d\psi + \omega_2 d\theta \wedge dr + \omega_3 d\theta \wedge d\psi$ for some functions ω_j , $j = 1, 2, 3$. Then

$$\int_{\mathcal{B}} d\omega = \int \partial_\theta \omega_1 d\theta dr d\psi + \int \partial_\psi \omega_2 d\psi d\theta dr + \int \partial_r \omega_3 dr d\theta d\psi, \quad (3.20)$$

with appropriate boundaries for the integrals.

Hence, the problem comes down to finding ω and evaluating the relevant asymptotic limits of ω_j . A natural choice for $\hat{\omega}$ is

$$\hat{\omega} = \sin \Psi \cos \Theta d\Psi \wedge d\Phi. \quad (3.21)$$

Now, to obtain the pullback $\omega = {}^*\hat{\omega}$, one needs to find the relation between coordinates (Θ, Φ, Ψ) on S^3 and (r, θ, ψ) on \mathcal{B} .

First we note that the Skyrme field U takes value in $SU(2)$. Now an element in $SU(2)$ of the form $\exp(-i a_j \tau_j)$ for some a_j can be written as

$$\exp(-i a_j \tau_j) = (\cos a) \mathbf{1} - i(\sin a) n_j \tau_j,$$

where $a_j = a n_j$ and $\mathbf{n} = (n_j)$ is a unit vector. In our case, $a_j = \pi \gamma_j$. The Taub-NUT

and Atiyah-Hitchin Skyrmsions [6] are of the form (2.14), with

$$\gamma_1 = f_1(r) \sin \theta \cos \psi, \quad \gamma_2 = f_2(r) \sin \theta \sin \psi, \quad \gamma_3 = f_3(r) \cos \theta,$$

for some functions $f_j(r)$. Thus,

$$U = \cos(\pi\kappa) \mathbf{1} - i \sin(\pi\kappa) \mathbf{n} \cdot \boldsymbol{\tau}, \quad (3.22)$$

where $\kappa = \sqrt{(f_1 n_1)^2 + (f_2 n_2)^2 + (f_3 n_3)^2}$ and $\mathbf{n} = (\sin \theta \cos \psi, \sin \theta \sin \psi, \cos \theta)$.

By comparing (3.18) and (3.22), one concludes that

$$\begin{aligned} \cos \Psi &= \cos(\pi\kappa), \\ \sin \Psi \cos \Theta &= \frac{\sin(\pi\kappa)}{\kappa} f_3 \cos \theta, \\ \sin \Psi \sin \Theta \cos \Phi &= \frac{\sin(\pi\kappa)}{\kappa} f_1 \sin \theta \cos \psi, \\ \sin \Psi \sin \Theta \sin \Phi &= \frac{\sin(\pi\kappa)}{\kappa} f_2 \sin \theta \sin \psi. \end{aligned}$$

Then choosing

$$\Psi = \pi\kappa$$

leads to

$$\begin{aligned} \cos \Theta &= \frac{f_3}{\kappa} \cos \theta \\ \sin \Theta \cos \Phi &= \frac{f_1}{\kappa} \sin \theta \cos \psi \\ \sin \Theta \sin \Phi &= \frac{f_2}{\kappa} \sin \theta \sin \psi. \end{aligned}$$

The last two equations give

$$\tan \Phi = \frac{f_2}{f_1} \tan \psi,$$

and thus

$$d\Phi = d\left(\frac{f_2}{f_1} \tan \psi\right) \frac{(f_1 \cos \psi \sin \theta)^2}{\kappa^2 - f_3^2 \cos^2 \theta}.$$

From the above relations, it follows that the pullback $\omega = {}^*\hat{\omega}$, with $\hat{\omega}$ given in (3.21), is given by

$$\omega = \frac{\pi \sin^2(\pi\kappa)}{\kappa} f_3 n_3 \frac{f_1^2 n_1^2}{f_1^2 n_1^2 + f_2^2 n_2^2} d\kappa \wedge d\left(\frac{f_2}{f_1} \tan \psi\right). \quad (3.23)$$

Finally, writing $\omega = \omega_1 dr \wedge d\psi + \omega_2 d\theta \wedge dr + \omega_3 d\theta \wedge d\psi$ yields

$$\omega_j = \beta \mu_j,$$

where

$$\begin{aligned}\beta &= \frac{\pi \sin^2(\pi \kappa)}{\kappa} f_3 n_3 \frac{f_1^2 n_1^2}{f_1^2 n_1^2 + f_2^2 n_2^2}, \\ \mu_1 &= \kappa_r \frac{f_2}{f_1} \sec^2 \psi - \kappa_\psi \left(\frac{f_2}{f_1} \right)_r \tan \psi, \\ \mu_2 &= \kappa_\theta \left(\frac{f_2}{f_1} \right)_r \tan \psi, \\ \mu_3 &= \kappa_\theta \frac{f_2}{f_1} \tan \psi.\end{aligned}$$

For the Taub-NUT Skymion, we have that

$$f_1 = f_2 = -\frac{r}{r+1}, \quad f_3 = \frac{r(r+2)}{(r+1)^2},$$

and the integral (3.20) becomes

$$\int \omega_1 \Big|_{\theta=0}^{\theta=\pi} dr d\psi + \int \omega_2 \Big|_{\psi=0}^{\psi=4\pi} d\theta dr + \int \omega_3 \Big|_{r=\infty}^{r=0} d\theta d\psi. \quad (3.24)$$

It follows that ω_2 is identically zero, and $\omega_3 \rightarrow 0$ as $r \rightarrow 0$ or $r \rightarrow \infty$. Thus the only nonzero contribution comes from the first term in (3.24). It can be shown that

$$\omega_1 \Big|_{\theta=0}^{\theta=\pi} = -2\pi \sin^2(\pi f_3) f_{3r}.$$

This yields the topological degree (3.19) for the Taub-NUT Skymion

$$D_{TN} = 2,$$

which is consistent with the result in [6] obtained by counting preimages.

For the Atiyah-Hitchin Skymion, the integral (3.20) becomes

$$\int d\omega = \int \omega_1 \Big|_{\theta=0}^{\theta=\pi} dr d\psi + \int \omega_2 \Big|_{\psi=0}^{\psi=2\pi} d\theta dr + \int \omega_3 \Big|_{r=\pi}^{r=\infty} d\theta d\psi. \quad (3.25)$$

However, we only know the asymptotic expressions for $f_j(r)$:

For large r ,

$$f_1 = f_2 = -\frac{r}{r-2}, \quad f_3 = \frac{r(r-4)}{(r-2)^2}, \quad (3.26)$$

and for r close to π ,

$$f_1 = \frac{\pi-r}{\pi} - 3, \quad f_2 = \frac{\pi-r}{\pi}, \quad f_3 = \frac{r-\pi}{r+\pi}. \quad (3.27)$$

Now it turns out that if we choose $\hat{\omega}$ in (3.21), the only nonzero contribution of the integral (3.25) comes from the first term which we are unable to integral as the expressions of $f_j(r)$ are not known. However, we can proceed with the same method by choosing a different $\hat{\omega}$ as

$$\hat{\omega}' = \frac{1}{2} \left(\Psi - \frac{1}{2} \sin 2\Psi \right) \sin \Theta \, d\Theta \wedge d\Phi, \quad (3.28)$$

where we have put the prime in to distinguish it from $\hat{\omega}$ in (3.21).

Then the pullback $\omega' = *\hat{\omega}'$ takes the form

$$\omega' = \frac{1}{2} \left(\frac{1}{2} \sin(2\pi\kappa) - \pi\kappa \right) \frac{f_1^2 n_1^2}{f_1^2 n_1^2 + f_2^2 n_2^2} d \left(\frac{f_3}{\kappa} \cos \theta \right) \wedge d \left(\frac{f_2}{f_1} \tan \psi \right).$$

Again, with $\omega' = \omega'_1 dr \wedge d\psi + \omega'_2 d\theta \wedge dr + \omega'_3 d\theta \wedge d\psi$, this results in

$$\omega'_j = \beta' \mu'_j,$$

where

$$\begin{aligned} \beta' &= \frac{1}{2} \left(\frac{1}{2} \sin(2\pi\kappa) - \pi\kappa \right) \frac{f_1^2 n_1^2}{f_1^2 n_1^2 + f_2^2 n_2^2}, \\ \mu'_1 &= \frac{f_2}{f_1} \left(\frac{f_3}{\kappa} \right)_r \cos \theta \sec^2 \psi + \left(\frac{f_2}{f_1} \right)_r \frac{\kappa_\psi}{\kappa^2} f_3 \cos \theta \tan \psi, \\ \mu'_2 &= \left(\frac{f_2}{f_1} \right)_r \left(\frac{\cos \theta}{\kappa} \right)_\theta f_3 \tan \psi, \\ \mu'_3 &= \frac{f_2}{f_1} \left(\frac{\cos \theta}{\kappa} \right)_\theta f_3 \sec^2 \psi. \end{aligned}$$

We note here that using $\hat{\omega}'$ in (3.28) we consistently obtain the topological degree for the Taub-NUT Skymion $D_{TN} = 2$. Only that this time the integrands in the first and second terms of (3.24) vanish, and only the third term gives nonzero contribution.

The same situation happens for the Atiyah-Hitchin Skymion, i.e. the integrands in the first and second terms of (3.25) vanish. Now, we can evaluate the last term using the asymptotic expressions (3.26) and (3.27).

As $r \rightarrow \infty$, one has that $(f_1, f_2, f_3) \rightarrow (-1, -1, 1)$ and $\kappa \rightarrow 1$. Then,

$$\beta' \rightarrow -\frac{\pi}{2} \cos^2 \psi.$$

Also, it can be shown that $\kappa_\theta \rightarrow 0$ as $r \rightarrow \infty$, thus

$$\mu'_3 \rightarrow -\sin \theta \sec^2 \psi.$$

Therefore

$$\lim_{r \rightarrow \infty} \omega'_3 = \frac{\pi}{2} \sin \theta.$$

Now as $r \rightarrow \pi$, we have $(f_1, f_2, f_3) \rightarrow (-3, 0, 0)$, which implies that $\lim_{r \rightarrow \pi} \omega'_3 = 0$. This gives the topological degree for the Atiyah-Hitchin Skymion

$$D_{AH} = -1.$$

The result is consistent with that in [6] up to a sign.

4 Relation to $SU(\infty)$ -Toda equation

In this last section we shall investigate the metric on the space where the Skymions live. Regardless of the gauge, the multi-Taub-NUT Skymions constructed in Section 2 are defined on the space of orbits \mathcal{B} of the Killing vector field $\frac{\partial}{\partial \phi}$. There is a natural metric on \mathcal{B} induced by the multi-Taub-NUT metric (2.4).

To find it, we consult a theorem by LeBrun [11], which states that any scalar-flat Kähler metric with a Killing symmetry preserving the Kähler form can be written in a particular form, and is determined by solutions of the $SU(\infty)$ -Toda equation

$$u_{xx} + u_{yy} + (e^u)_{tt} = 0, \tag{4.1}$$

and its linearisation

$$W_{xx} + W_{yy} + (We^u)_{tt} = 0. \tag{4.2}$$

A multi-Taub-NUT metric is hyperKähler, hence scalar-flat Kähler. Also, as we will see shortly, the Killing vector field $\frac{\partial}{\partial \phi}$ preserves a Kähler form. Then, the metric g (2.4) and the Kähler form ω can be written in the LeBrun ansatz as

$$g = Wh + \frac{1}{W}(d\phi + \lambda)^2, \quad h = e^u(dx^2 + dy^2) + dt^2, \tag{4.3}$$

$$\omega = We^u dx \wedge dy + (d\phi + \lambda) \wedge dt \quad (4.4)$$

where $\{x, y, t\}$ are coordinates on the space of orbits, and (u, W) , λ and h are functions, a one-form and a metric on the space of orbits, respectively. The function $u = u(x, y, t)$ necessarily satisfies the $SU(\infty)$ -Toda equation (4.1) and W satisfies the so-called monopole equation (4.2).

4.1 Metric on the space of orbits

We shall now find the induced metric h on \mathcal{B} and identify its associated solution of the $SU(\infty)$ -Toda equation (4.1). For convenience, in this section let us consider the multi-Taub-NUT metric in the cylindrical coordinates:

$$g = V(d\rho^2 + \rho^2 d\phi^2 + dz^2) + V^{-1}(d\psi + \alpha)^2, \quad (4.5)$$

where we recall that $V(\rho, z)$ is a solution of the Laplace's equation

$$\rho V_{zz} + (\rho V_\rho)_\rho = 0, \quad (4.6)$$

and $\alpha = \hat{\alpha}(\rho, z)d\phi$. The relation $d\alpha = *_3 dV$ implies that

$$\hat{\alpha}_z = -\rho V_\rho, \quad \hat{\alpha}_\rho = \rho V_z. \quad (4.7)$$

Proposition 4.1 *Let g be the axially symmetric multi-Taub-NUT metric given by (4.5) with*

$$V = 1 + \sum_{n=1}^N \frac{1}{\sqrt{\rho^2 + (z - z_n)^2}} \quad \text{and} \quad \alpha = \sum_{n=1}^N \frac{z - z_n}{\sqrt{\rho^2 + (z - z_n)^2}} d\phi. \quad (4.8)$$

Then the Einstein-Weyl metric on the space of orbits of the Killing vector field $K = \frac{\partial}{\partial \phi}$ is of the form $h = e^u(dx^2 + dy^2) + dt^2$, where u is the solution to the $SU(\infty)$ -Toda equation (4.1) which is independent of y and given implicitly by $u(x, t) = \ln(\rho^2)$ and

$$x = -z + \ln \left(\frac{\rho^N}{\prod_{n=1}^N \left(z - z_n + \sqrt{\rho^2 + (z - z_n)^2} \right)} \right), \quad (4.9)$$

$$t = \frac{\rho^2}{2} + \sum_{n=1}^N \sqrt{\rho^2 + (z - z_n)^2}. \quad (4.10)$$

Proof First it can be shown that the Killing symmetry generated by $K = \frac{\partial}{\partial \phi}$ preserves a Kähler structure of (4.5). The complex structure I is defined by the holomorphic basis of one-forms

$$\begin{aligned} e^1 &= \frac{1}{\sqrt{V}}(d\psi + \alpha) + i\sqrt{V}dz \\ e^2 &= \sqrt{V}d\rho - i\sqrt{V}\rho d\phi, \end{aligned}$$

and the Kähler form given by

$$\omega = (d\psi + \alpha) \wedge dz + \rho V d\phi \wedge d\rho. \quad (4.11)$$

It then follows from (4.7) that the Lie derivative $\mathcal{L}_K \omega = d(K \lrcorner \omega)$ vanishes.

Now by the theorem of LeBrun, there exists a local coordinate system $\{\phi, x, y, t\}$ such that the metric (4.5) takes the form (4.3). To see this, we note that any metric with a Killing vector field $K = \frac{\partial}{\partial \phi}$ necessarily takes the form of the first equation in (4.3), where $\frac{1}{W} = g(K, K)$ and h is a metric on \mathcal{B} - the three-dimensional space of orbits of K . The vanishing of $\mathcal{L}_K \omega = d(K \lrcorner \omega)$ implies that

$$K \lrcorner \omega = dt \quad (4.12)$$

for some function t on the \mathcal{B} . Next, we can use isothermal coordinates x, y on the orthogonal complement of the space spanned by K and $I(K)$, together with t , as local coordinates on \mathcal{B} . This will result in the form (4.3). The equations (4.2) and (4.1) come from the Kähler condition and the scalar-flat condition, respectively.

To obtain the metric h of (4.3), first we complete the square and rearrange (4.5). This gives

$$\begin{aligned} h &= \rho^2 d\psi^2 + (V^2 \rho^2 + \hat{\alpha}^2)(d\rho^2 + dz^2), \\ W &= \frac{V}{V^2 \rho^2 + \hat{\alpha}^2} \quad \text{and} \quad \lambda = \frac{\hat{\alpha}}{V^2 \rho^2 + \hat{\alpha}^2} d\psi \end{aligned} \quad (4.13)$$

Then from (4.12), we have

$$dt = \rho V d\rho + \hat{\alpha} dz. \quad (4.14)$$

The other two coordinates, x and y , can be chosen such that

$$dx = \rho^{-1} \hat{\alpha} d\rho - V dz, \quad dy = d\psi, \quad (4.15)$$

so that h is of the form (4.3). Then we have

$$e^u = \rho^2. \quad (4.16)$$

Equation (4.16) gives the solution u of the $SU(\infty)$ -Toda equation implicitly. It can be readily verified using the chain rule that $u = \ln(\rho^2)$ indeed satisfies (4.1). By integrating (4.14, 4.15), one obtains (4.9, 4.10).

□

4.2 Limit $N \rightarrow \infty$

As N becomes large, the function V of the multi-Taub-NUT metric (4.5) approaches the Ooguri-Vafa limit [14], such that when $\rho \rightarrow \infty$, V can be approximated by

$$V = -\ln(\rho^2).$$

Then, using the same procedure as in the proof of Proposition 4.1, one obtains the Einstein-Weyl metric h on the space of orbits of the vector field $K = \frac{\partial}{\partial \phi}$, of the form $h = e^u(dx^2 + dy^2) + dt^2$, where the solution to the $SU(\infty)$ -Toda equation u is given implicitly by $u(x, t) = \ln(\rho^2)$ and

$$x = z \ln(\rho^2), \quad t = z^2 - \frac{1}{2}\rho^2(\ln(\rho^2) - 1). \quad (4.17)$$

The relations (4.17) yield

$$u^2(u - 1)e^u + 2(t - x^2) = 0,$$

which implies that u is a solution which is constant on the cylinder $x^2 - t = k$, where k is a constant.

We note here that solutions of the $SU(\infty)$ -Toda equation constant on surfaces have been studied previously in [18] and [7], where the surfaces are central ellipsoids and planes, respectively.

4.3 More explicit expressions

To obtain an explicit expression for the solution $u(x, t)$ in Proposition 4.1, one needs to know $\rho(x, t)$ explicitly. In principle, this can be achieved by inverting the expressions (4.9, 4.10). However, it turns out to be impossible even for the case $N = 1$ of the Taub-NUT

metric. Therefore, to obtain a more explicit description of the corresponding solution to the $SU(\infty)$ -Toda equation and thus the Einstein-Weyl metric, we proceed analogously to [19] as follows.

We are guided by the fact that the solution u of (4.1) is independent of y . Such a solution belongs to the class of solutions which can be obtained from axisymmetric solutions of the Laplace's equation, considered by Ward in [19].¹ The procedure is based on a transformation of variables as described below.

The multi-Taub-NUT metric (4.5) is governed by the axially symmetric solution V (4.8) of the Laplace's equation, satisfying

$$\rho V_{zz} + (\rho V_\rho)_\rho = 0. \quad (4.18)$$

One notes that the function \widehat{V}

$$\widehat{V} = \sum_{n=1}^N \ln \left[(z - z_n) + \sqrt{\rho^2 + (z - z_n)^2} \right], \quad (4.19)$$

given by $V = 1 + \widehat{V}_z$, is also a solution of (4.18). Now, define new variables (ξ, τ) by

$$\xi = \widehat{V}_z, \quad \tau = \frac{\hat{\alpha}}{2}, \quad (4.20)$$

where the function $\hat{\alpha}$ is given in (4.8) (notice that $\hat{\alpha} = N - \rho \widehat{V}_\rho$). Inverting (4.20) to obtain $\rho(\xi, \tau)$, it follows that if we let

$$u = \ln \left(\frac{\rho^2}{4} \right), \quad (4.21)$$

then u satisfies the reduced $SU(\infty)$ -Toda equation

$$u_{\xi\xi} + (e^u)_{\tau\tau} = 0. \quad (4.22)$$

Our aim now is to obtain an expression, as close to being explicit as possible, for $\rho(\xi, \tau)$ for \widehat{V} in (4.19). The change of variables (4.20) gives

$$\xi = \sum_{n=1}^N y_n, \quad 2\tau = z\xi - \sum_{n=1}^N z_n y_n, \quad (4.23)$$

¹The procedure in [19] was in fact formulated in the $2+1$ dimension, with the wave equation in place of the Laplace's equation. However, it can be readily adapted to the Euclidean signature.

where $y_n = \frac{1}{\sqrt{\rho^2 + (z - z_n)^2}}$.

In principle, one could use the equations (4.23) together with the expression for $y_n(\rho, z)$ to obtain $\rho^2(\xi, \tau)$. The idea is best illustrated through the following examples of the Taub-NUT metric ($N = 1$) and the multi-Taub-NUT metrics with $N = 2$ and $N = 3$.

First, let us consider the Taub-NUT metric, where $N = 1$, $z_n = 0$. The equations (4.23) becomes

$$\xi = y_1, \quad 2\tau = z\xi.$$

Here, we simply write $z = 2\tau/\xi$, and from $y_1 = \frac{1}{\rho^2 + z^2}$ obtain

$$\rho^2 = \frac{1}{\xi^2} - z^2 = \frac{1 - 4\tau^2}{\xi^2}.$$

Thus the corresponding solution of the $SU(\infty)$ -Toda equation, $u = \ln\left(\frac{\rho^2}{4}\right)$, is given by

$$u = \ln\left(\frac{1/4 - \tau^2}{\xi^2}\right).$$

Next, for the multi-Taub-NUT $N = 2$, let us consider a case where $z_1 = 0$ and $z_2 = 1$ for simplicity. We now have

$$\xi = y_1 + y_2, \quad 2\tau = z\xi - y_2.$$

This is a linear system from which we can easily write y_1, y_2 as functions of ξ, τ and z :

$$y_1 = \xi + 2\tau - z\xi, \quad y_2 = z\xi - 2\tau.$$

Now, using the expressions of y_1 and y_2 , one obtains

$$\rho^2 + z^2 = \frac{1}{(\xi + 2\tau - z\xi)^2}, \quad \rho^2 + (z - 1)^2 = \frac{1}{(z\xi - 2\tau)^2}. \quad (4.24)$$

The difference of the two equations in (4.24) gives a fifth order polynomial equation for z in terms of ξ, τ :

$$(\xi + 2\tau - z\xi)^2(z\xi - 2\tau)^2(2z - 1) = 2\xi(z\xi - 2\tau) - \xi^2$$

The root of the polynomial which satisfies the system (4.24) will give the desired expression for $z(\xi, \tau)$, which in turn yields $u(\xi, \tau) = \ln \left(\frac{\rho^2}{4} \right)$ from

$$\rho^2 = \frac{1}{(\xi + 2\tau - z\xi)^2 - z^2}.$$

Lastly, just to see how the computation develops as N becomes larger, let us consider the case of the multi-Taub-NUT metric $N = 3$. For simplicity, let $z_1 = 0, z_2 = 1$ and $z_3 = -1$. Then (4.23) becomes

$$\xi = y_1 + y_2 + y_3, \quad 2\tau = z\xi - y_2 + y_3. \quad (4.25)$$

To proceed similarly to the $N = 2$ case, one can start by writing $y_3 = \frac{y_2}{\sqrt{1 + 4zy_2^2}}$. The system (4.25) is no longer a linear system of equations for y_1, y_2 . Thus, unlike the case $N = 2$, to get expressions for $y_1(z, \xi, \tau)$, $y_2(z, \xi, \tau)$, one needs to solve a polynomial equation. For example, one can find y_2 from

$$(2\tau - z\xi + y_2)^2(1 + 4zy_2^2) = y_2^2$$

and subsequently obtain y_1 .

With $y_1(z, \xi, \tau)$, $y_2(z, \xi, \tau)$, one can proceed similarly to the case $N = 2$ to get $z(\xi, \tau)$ from

$$\rho^2 + z^2 = \frac{1}{y_1^2(z, \xi, \tau)}, \quad \rho^2 + (z - 1)^2 = \frac{1}{y_2^2(z, \xi, \tau)}.$$

The difference of the two equations above will give a high order polynomial for z . The root $z(\xi, \tau)$ would then yield $\rho^2(\xi, \tau)$ via $\rho^2 = \frac{1}{y_1^2 - z^2}$ as before.

Let us conclude this section by writing down the metric h (4.13)

$$h = \rho^2 d\psi^2 + (V^2 \rho^2 + \hat{\alpha}^2)(d\rho^2 + dz^2)$$

on the space of orbits \mathcal{B} . We shall use the coordinates (ξ, y, τ) , where (ξ, τ) are defined in (4.20), as coordinates on \mathcal{B} , given by

$$\xi = V - 1, \quad y = \psi, \quad \tau = \frac{\hat{\alpha}}{2},$$

and let $e^u = \rho^2/4$. In these coordinates, h becomes

$$h/4 = e^u (\mathcal{H} d\xi^2 + dy^2) + \mathcal{H} d\tau^2,$$

where $\mathcal{H}(\xi, \tau)$ is a function given by

$$\mathcal{H} = \frac{e^u(1 + \xi)^2 + \tau^2}{e^u(\xi_\rho^2 + \xi_z^2)},$$

and ξ_ρ, ξ_z denote $\frac{\partial \xi}{\partial \rho}$ and $\frac{\partial \xi}{\partial z}$, respectively.

The difficulty in writing h explicitly in the coordinates (ξ, y, τ) comes down to writing ξ_ρ, ξ_z as functions of (ξ, τ) . A direct computation shows that this requires an expression for $z(\xi, \tau)$, which we have obtained only for the Taub-NUT case, where

$$\mathcal{H} = \frac{e^u(1 + \xi)^2 + \tau^2}{e^u \xi^4}.$$

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